NOTE OF ELEMENTARY ANALYSIS II

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1. Riemann Integrals

Notation 1.1. .

(i) : *All functions* $f, g, h...$ *are bounded real valued functions defined on* $[a, b]$. *And* $m \leq f \leq M$.

(ii) : $\mathcal{P} : a = x_0 < x_1 < .... < x_n = b$ *denotes a partition on* $[a, b]$; $\Delta x_i = x_i - x_{i-1}$ *and* $\|\mathcal{P}\| = \max \Delta x_i$.

(iii) : $M_i(f, \mathcal{P}) := \sup\{f(x) : x \in [x_{i-1}, x_i\}$; $m_i(f, \mathcal{P}) := \inf\{f(x) : x \in [x_{i-1}, x_i\}$. *And* $\omega_i(f, \mathcal{P}) = M_i(f, \mathcal{P}) - m_i(f, \mathcal{P})$.

(iv) : $U(f, \mathcal{P}) := \sum M_i(f, \mathcal{P})\Delta x_i$; $L(f, \mathcal{P}) := \sum m_i(f, \mathcal{P})\Delta x_i$.

(v) : $\mathcal{R}(f, \mathcal{P}, \{\xi_i\}) := \sum f(\xi_i)\Delta x_i$, *where* $\xi_i \in [x_{i-1}, x_i]$.

(vi) : $\mathcal{R}[a, b]$ *is the class of all Riemann integral functions on* $[a, b]$.

Definition 1.2. *We say that the Riemann sum* $\mathcal{R}(f, \mathcal{P}, \{\xi_i\})$ *converges to a number* A *as* $\|\mathcal{P}\| \to 0$ *if for any* $\varepsilon > 0$, *there is* $\delta > 0$ *such that*

$$|A - \mathcal{R}(f, \mathcal{P}, \{\xi_i\})| < \varepsilon$$

for any $\xi_i \in [x_{i-1}, x_i]$ *whenever* $\|\mathcal{P}\| < \delta$.

Theorem 1.3. $f \in \mathcal{R}[a, b]$ *if and only if for any* $\varepsilon > 0$, *there is a partition* $\mathcal{P}$ *such that* $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$.

Lemma 1.4. $f \in \mathcal{R}[a, b]$ *if and only if for any* $\varepsilon > 0$, *there is* $\delta > 0$ *such that* $U(f, \mathcal{P}) - L(f, \mathcal{P}) < \varepsilon$ *whenever* $\|\mathcal{P}\| < \delta$.

Proof. The converse follows from Theorem 1.3.
Assume that f is integrable over $[a, b]$. Let $\varepsilon > 0$. Then there is a partition $\mathcal{Q} : a = y_0 < ... < y_l = b$ on $[a, b]$ such that $U(f, Q) - L(f, Q) < \varepsilon$. Now take $0 < \delta < \varepsilon/l$. Suppose that $\mathcal{P} : a = x_0 < ... < x_n = b$ with $\|\mathcal{P}\| < \delta$. Then we have

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) = I + II$$

where

$$I = \sum_{i:Q \cap (x_{i-1}, x_i) = \emptyset} \omega_i(f, \mathcal{P})\Delta x_i;$$

and

$$II = \sum_{i:Q \cap (x_{i-1}, x_i) \neq \emptyset} \omega_i(f, \mathcal{P})\Delta x_i$$

Notice that we have

$$I \leq U(f, \mathcal{Q}) - L(f, \mathcal{Q}) < \varepsilon$$

and

$$II \leq (M - m) \sum_{i:Q \cap (x_{i-1}, x_i) \neq \emptyset} \Delta x_i \leq (M - m) \cdot l \cdot \frac{\varepsilon}{l} = (M - m)\varepsilon.$$

The proof is finished. □

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Theorem 1.5. *$f \in \mathcal{R}[a,b]$ if and only if the Riemann sum $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ is convergent. In this case, $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ converges to $\displaystyle\int_a^b f(x)dx$ as $\|\mathcal{P}\| \to 0$.*

Proof. For the proof $(\Rightarrow)$: we first note that we always have

$$L(f,\mathcal{P}) \leq \mathcal{R}(f,\mathcal{P},\{\xi_i\}) \leq U(f,\mathcal{P})$$

and

$$L(f,\mathcal{P}) \leq \int_a^b f(x)dx \leq U(f,\mathcal{P})$$

for any $\xi_i \in [x_{i-1}, x_i]$ and for all partition $\mathcal{P}$.

Now let $\varepsilon > 0$. Lemma 1.4 gives $\delta > 0$ such that $U(f,\mathcal{P}) - L(f,\mathcal{P}) < \varepsilon$ as $\|\mathcal{P}\| < \delta$. Then we have

$$|\int_a^b f(x)dx - \mathcal{R}(f,\mathcal{P},\{\xi_i\})| < \varepsilon$$

as $\|\mathcal{P}\| < \delta$. The necessary part is proved and $\mathcal{R}(f,\mathcal{P},\{\xi_i\})$ converges to $\displaystyle\int_a^b f(x)dx$.

For $(\Leftarrow)$: there exists a number A such that for any $\varepsilon > 0$, there is $\delta > 0$, we have

$$A - \varepsilon < \mathcal{R}(f,\mathcal{P},\{\xi_i\}) < A + \varepsilon$$

for any partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$ and $\xi_i \in [x_{i-1}, x_i]$.

Now fix a partition $\mathcal{P}$ with $\|\mathcal{P}\| < \delta$. Then for each $[x_{i-1}, x_i]$, choose $\xi_i \in [x_{i-1}, x_i]$ such that $M_i(f,\mathcal{P}) - \varepsilon \leq f(\xi_i)$. This implies that we have

$$U(f,\mathcal{P}) - \varepsilon(b-a) \leq \mathcal{R}(f,\mathcal{P},\{\xi_i\}) < A + \varepsilon.$$

So we have shown that for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that

$$\overline{\int_a^b} f(x)dx \leq U(f,\mathcal{P}) \leq A + \varepsilon(1+b-a). \tag{1.1}$$

By considering $-f$, note that the Riemann sum of $-f$ will converge to $-A$. The inequality 1.1 will imply that for any $\varepsilon > 0$, there is a partition $\mathcal{P}$ such that

$$A - \varepsilon(1+b-a) \leq \underline{\int_a^b} f(x)dx \leq \overline{\int_a^b} f(x)dx \leq A + \varepsilon(1+b-a).$$

The proof is finished. □

Theorem 1.6. *Let $f \in \mathcal{R}[c,d]$ and let $\phi : [a,b] \longrightarrow [c,d]$ be a strictly increasing C^1 function with $f(a) = c$ and $f(b) = d$.*
Then $f \circ \phi \in \mathcal{R}[a,b]$, moreover, we have

$$\int_c^d f(x)dx = \int_a^b f(\phi(t))\phi'(t)dt.$$

Proof. Let $A = \int_c^d f(x)dx$. By Theorem 1.5, we need to show that for all $\varepsilon > 0$, there is $\delta > 0$ such that

$$|A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| < \varepsilon$$

for all $\xi_k \in [t_{k-1}, t_k]$ whenever $\mathcal{Q} : a = t_0 < ... < t_m = b$ with $\|\mathcal{Q}\| < \delta$.

Now let $\varepsilon > 0$. Then by Lemma 1.4 and Theorem 1.5, there is $\delta_1 > 0$ such that

$$|A - \sum f(\eta_k)\triangle x_k| < \varepsilon \tag{1.2}$$

and

$$\sum \omega_k(f,\mathcal{P})\triangle x_k < \varepsilon \tag{1.3}$$

for all $\eta_k \in [x_{k-1}, x_k]$ whenever $\mathcal{P} : c = x_0 < ... < x_m = d$ with $\|\mathcal{P}\| < \delta_1$.
Now put $x = \phi(t)$ for $t \in [a, b]$.
Now since ϕ and ϕ' are continuous on $[a, b]$, there is $\delta > 0$ such that $|\phi(t) - \phi(t')| < \delta_1$ and $|\phi'(t) - \phi'(t')| < \varepsilon$ for all t, t' $in[a, b]$ with $|t - t'| < \delta$.
Now let $\mathcal{Q} : a = t_0 < ... < t_m = b$ with $\|\mathcal{Q}\| < \delta$. If we put $x_k = \phi(t_k)$, then $\mathcal{P} : c = x_0 < .... < x_m = d$ is a partition on $[c, d]$ with $\|\mathcal{P}\| < \delta_1$ because ϕ is strictly increasing.
Note that the Mean Value Theorem implies that for each $[t_{k-1}, t_k]$, there is $\xi_k^* \in (t_{k-1}, t_k)$ such that

$$\triangle x_k = \phi(t_k) - \phi(t_{k-1}) = \phi'(\xi_k^*)\triangle t_k.$$

This yields that

$$|\triangle x_k - \phi'(\xi_k)\triangle t_k| < \varepsilon \triangle t_k \tag{1.4}$$

for any $\xi_k \in [t_{k-1}, t_k]$ for all $k = 1, ..., m$ because of the choice of δ.
Now for any $\xi_k \in [t_{k-1}, t_k]$, we have

$$\begin{aligned} |A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \leq{}& |A - \sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k| \\ &+ |\sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k - \sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k| \\ &+ |\sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \end{aligned} \tag{1.5}$$

Notice that inequality 1.2 implies that

$$|A - \sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k| = |A - \sum f(\phi(\xi_k^*))\triangle x_k| < \varepsilon.$$

Also, since we have $|\phi'(\xi_k^*) - \phi'(\xi_k)| < \varepsilon$ for all $k = 1, .., m$, we have

$$|\sum f(\phi(\xi_k^*))\phi'(\xi_k^*)\triangle t_k - \sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k| \leq M(b - a)\varepsilon$$

where $|f(x)| \leq M$ for all $x \in [c, d]$.
On the other hand, by using inequality 1.4 we have

$$|\phi'(\xi_k)\triangle t_k| \leq \triangle x_k + \varepsilon \triangle t_k$$

for all k. This, together with inequality 1.3 imply that

$$\begin{aligned} &|\sum f(\phi(\xi_k^*))\phi'(\xi_k)\triangle t_k - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \\ &\leq \sum \omega_k(f, \mathcal{P})|\phi'(\xi_k)\triangle t_k| \ (\because \phi(\xi_k^*), \phi(\xi_k) \in [x_{k-1}, x_k]) \\ &\leq \sum \omega_k(f, \mathcal{P})(\triangle x_k + \varepsilon \triangle t_k) \\ &\leq \varepsilon + 2M(b - a)\varepsilon. \end{aligned}$$

Finally by inequality 1.5, we have

$$|A - \sum f(\phi(\xi_k))\phi'(\xi_k)\triangle t_k| \leq \varepsilon + M(b - a)\varepsilon + \varepsilon + 2M(b - a)\varepsilon.$$

The proof is finished. □

Example 1.7. *Define (*formally*) an improper integral* $\Gamma(s)$ *(called the* Γ*-function) as follows:*

$$\Gamma(s) := \int_0^\infty x^{s-1}e^{-x}dx$$

for $s \in \mathbb{R}$*. Then* $\Gamma(s)$ *is convergent if and only if* $s > 0$*.*

Proof. Put $I(s) := \int_0^1 x^{s-1}e^{-x}dx$ and $II(s) := \int_1^\infty x^{s-1}e^{-x}dx$. We first claim that the integral $II(s)$ is convergent for all $s \in \mathbb{R}$.
In fact, if we fix $s \in \mathbb{R}$, then we have

$$\lim_{x\to\infty} \frac{x^{s-1}}{e^{x/2}} = 0.$$

So there is $M > 1$ such that $\frac{x^{s-1}}{e^{x/2}} \leq 1$ for all $x \geq M$. Thus we have

$$0 \leq \int_M^\infty x^{s-1}e^{-x}dx \leq \int_M^\infty e^{-x/2}dx < \infty.$$

Therefore we need to show that the integral $I(s)$ is convergent if and only if $s > 0$.
Note that for $0 < \eta < 1$, we have

$$0 \leq \int_\eta^1 x^{s-1}e^{-x}dx \leq \int_\eta^1 x^{s-1}dx = \begin{cases} \frac{1}{s}(1-\eta^s) & \text{if } s-1 \neq -1; \\ -\ln\eta & \text{otherwise .} \end{cases}$$

Thus the integral $I(s) = \lim\limits_{\eta\to 0+} \int_\eta^1 x^{s-1}e^{-x}dx$ is convergent if $s > 0$.
Conversely, we also have

$$\int_\eta^1 x^{s-1}e^{-x}dx \geq e^{-1}\int_\eta^1 x^{s-1}dx = \begin{cases} \frac{e^{-1}}{s}(1-\eta^s) & \text{if } s-1 \neq -1; \\ -e^{-1}\ln\eta & \text{otherwise .} \end{cases}$$

So if $s \leq 0$, then $\int_\eta^1 x^{s-1}e^{-x}dx$ is divergent as $\eta \to 0+$. The result follows. □

2. Uniform Convergence of a Sequence of Differentiable Functions

Proposition 2.1. *Let $f_n : (a,b) \longrightarrow \mathbb{R}$ be a sequence of functions. Assume that it satisfies the following conditions:*

(i) : $f_n(x)$ point-wise converges to a function $f(x)$ on (a,b);
(ii) : each f_n is a C^1 function on (a,b);
(iii) : $f_n' \to g$ uniformly on (a,b).

Then f is a C^1-function on (a,b) with $f' = g$.

Proof. Fix $c \in (a,b)$. Then for each x with $c < x < b$ (similarly, we can prove it in the same way as $a < x < c$), the Fundamental Theorem of Calculus implies that

$$f_n(x) = \int_c^x f'(t)dt.$$

Since $f_n' \to g$ uniformly on (a,b), we see that

$$\int_c^x f_n'(t)dt \longrightarrow \int_c^x g(t)dt.$$

This gives

$$f(x) = \int_c^x g(t)dt. \qquad (2.1)$$

for all $x \in (c,b)$. On the other hand, g is continuous on (a,b) since each f_n' is continuous and $f_n' \to g$ uniformly on (a,b). Equation 2.1 will tell us that f' exists and $f' = g$ on (c,b). The proof is finished. □

Proposition 2.2. *Let (f_n) be a sequence of differentiable functions defined on (a,b). Assume that*

(i): there is a point $c \in (a,b)$ such that $\lim f_n(c)$ exists;
(ii): f_n' converges uniformly to a function g on (a,b).

Then

(a): f_n converges uniformly to a function f on (a,b);
(b): f is differentiable on (a,b) and $f'=g$.

Proof. For Part (a), we will make use the Cauchy theorem.
Let $\varepsilon>0$. Then by the assumptions (i) and (ii), there is a positive integer N such that

$$|f_m(c)-f_n(c)|<\varepsilon \quad \text{and} \quad |f_m'(x)-f_n'(x)|<\varepsilon$$

for all $m,n\geq N$ and for all $x\in(a,b)$. Now fix $c<x<b$ and $m,n\geq N$. To apply the Mean Value Theorem for f_m-f_n on (c,x), then there is a point ξ between c and x such that

$$f_m(x)-f_n(x)=f_m(c)-f_n(c)+(f_m'(\xi)-f_n'(\xi))(x-c). \tag{2.2}$$

This implies that

$$|f_m(x)-f_n(x)|\leq|f_m(c)-f_n(c)|+|f_m'(\xi)-f_n'(\xi)||x-c|<\varepsilon+(b-a)\varepsilon$$

for all $m,n\geq N$ and for all $x\in(c,b)$. Similarly, when $x\in(a,c)$, we also have

$$|f_m(x)-f_n(x)|<\varepsilon+(b-a)\varepsilon.$$

So Part (a) follows.
Let f be the uniform limit of (f_n) on (a,b)
For Part (b), we fix $u\in(a,b)$. We are going to show

$$\lim_{x\to u}\frac{f(x)-f(u)}{x-u}=g(u).$$

Let $\varepsilon>0$. Since $f_n\to f$ and $f'\to g$ both are uniformly convergent on (a,b). Then there is $N\in\mathbb{N}$ such that

$$|f_m(x)-f_n(x)|<\varepsilon \quad \text{and} \quad |f_m'(x)-f_n'(x)|<\varepsilon \tag{2.3}$$

for all $m,n\geq N$ and for all $x\in(a,b)$
Note that for all $m\geq N$ and $x\in(a,b)\setminus\{u\}$, applying the Mean value Theorem for f_m-f_N as before, we have

$$\frac{f_m(x)-f_N(x)}{x-u}=\frac{f_m(u)-f_N(u)}{x-u}+(f_m'(\xi)-f_N'(\xi))$$

for some ξ between u and x.
So Eq.2.3 implies that

$$|\frac{f_m(x)-f_m(u)}{x-u}-\frac{f_N(x)-f_N(u)}{x-u}|\leq\varepsilon \tag{2.4}$$

for all $m\geq N$ and for all $x\in(a,b)$ with $x\neq u$.
Taking $m\to\infty$ in Eq.2.4, we have

$$|\frac{f(x)-f(u)}{x-u}-\frac{f_N(x)-f_N(u)}{x-u}|\leq\varepsilon.$$

Hence we have

$$\begin{aligned}|\frac{f(x)-f(u)}{x-u}-f_N'(u)|&\leq|\frac{f(x)-f(u)}{x-c}-\frac{f_N(x)-f_N(u)}{x-u}|+|\frac{f_N(x)-f_N(u)}{x-u}-f_N'(u)|\\&\leq\varepsilon+|\frac{f_N(x)-f_N(u)}{x-u}-f_N'(u)|.\end{aligned}$$

So if we can take $0<\delta$ such that $|\frac{f_N(x)-f_N(u)}{x-u}-f_N'(u)|<\varepsilon$ for $0<|x-u|<\delta$, then we have

$$|\frac{f(x)-f(u)}{x-u}-f_N'(u)|\leq 2\varepsilon \tag{2.5}$$

for $0 < |x-u| < \delta$. On the other hand, by the choice of N, we have $|f'_m(y) - f'_N(y)| < \varepsilon$ for all $y \in (a,b)$ and $m \geq N$. So we have $|g(u) - f'_N(u)| \leq \varepsilon$. This together with Eq.2.5 give

$$|\frac{f(x)-f(u)}{x-u} - g(u)| \leq 3\varepsilon$$

as $0 < |x-u| < \delta$, that is we have

$$\lim_{x \to u} \frac{f(x)-f(u)}{x-u} = g(u).$$

The proof is finished. □

Remark 2.3. *The uniform convergence assumption of (f'_n) in Propositions 2.1 and 2.2 is essential.*

Example 2.4. *Let $f_n(x) := \tan^{-1} nx$ for $x \in (-1,1)$. Then we have*

$$f(x) := \lim_n \tan^{-1} nx = \begin{cases} \pi/2 & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ -\pi/2 & \text{if } x < 0. \end{cases}$$

Also $g(x) := \lim_n f'_n(x) = \lim_n 1/(1+n^2x^2) = 0$ for all $x \in (-1,1)$. So Propositions 2.1 and 2.2 does not hold. Note that (f'_n) does not converge uniformly to g on $(-1,1)$.

3. Absolutely convergent series

Throughout this section, let (a_n) be a sequence of complex numbers.

Definition 3.1. *We say that a series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} |a_n| < \infty$.*

Also a convergent series $\sum_{n=1}^{\infty} a_n$ is said to be conditionally convergent if it is not absolute convergent.

Example 3.2. Important Example : *The series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^\alpha}$ is conditionally convergent when* $0 < \alpha \leq 1$.

This example shows us that a convergent improper integral may fail to the absolute convergence or square integrable property.

For instance, if we consider the function $f : [1,\infty) \longrightarrow \mathbb{R}$ given by

$$f(x) = \frac{(-1)^{n+1}}{n^\alpha} \quad \text{if} \quad n \leq x < n+1.$$

If $\alpha = 1/2$, then $\int_1^\infty f(x)dx$ is convergent but it is neither absolutely convergent nor square integrable.

Notation 3.3. *Let $\sigma : \{1,2...\} \longrightarrow \{1,2....\}$ be a bijection. A formal series $\sum_{n=1}^{\infty} a_{\sigma(n)}$ is called an rearrangement of $\sum_{n=1}^{\infty} a_n$.*

Example 3.4. *In this example, we are going to show that there is an rearrangement of the series* $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}$ *is divergent although the original series is convergent. In fact, it is conditionally convergent.*
We first notice that the series $\sum_i \frac{1}{2i-1}$ *diverges to infinity. Thus for each* $M > 0$*, there is a positive integer* N *such that*

$$\sum_{i=1}^{n} \frac{1}{2i-1} \geq M \qquad \cdots\cdots\cdots (*)$$

for all $n \geq N$*. Then there is* $N_1 \in \mathbb{N}$ *such that*

$$\sum_{i=1}^{N_1} \frac{1}{2i-1} - \frac{1}{2} > 1.$$

By using $(*)$ *again, there is a positive integer* N_2 *with* $N_1 < N_2$ *such that*

$$\sum_{i=1}^{N_1} \frac{1}{2i-1} - \frac{1}{2} + \sum_{N_1 < i \leq N_2} \frac{1}{2i-1} - \frac{1}{4} > 2.$$

To repeat the same procedure, we can find a positive integers subsequence (N_k) *such that*

$$\sum_{i=1}^{N_1} \frac{1}{2i-1} - \frac{1}{2} + \sum_{N_1 < i \leq N_2} \frac{1}{2i-1} - \frac{1}{4} + \cdots\cdots\cdots - \sum_{N_{k-1} < i \leq N_k} \frac{1}{2i-1} - \frac{1}{2k} > k$$

for all positive integers k*. So if we let* $a_n = \frac{(-1)^{n+1}}{n}$*, then one can find a bijection* $\sigma : \mathbb{N} \to \mathbb{N}$ *such that the series* $\sum_{i=1}^{\infty} a_{\sigma(i)}$ *is an rearrangement of the series* $\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i}$ *and diverges to infinity. The proof is finished.*

Theorem 3.5. *Let* $\sum_{n=1}^{\infty} a_n$ *be an absolutely convergent series. Then for any rearrangement* $\sum_{n=1}^{\infty} a_{\sigma(n)}$ *is also absolutely convergent. Moreover, we have* $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_{\sigma(n)}$*.*

Proof. Let $\sigma : \{1, 2...\} \longrightarrow \{1, 2...\}$ be a bijection as before.
We first claim that $\sum_n a_{\sigma(n)}$ is also absolutely convergent.
Let $\varepsilon > 0$. Since $\sum_n |a_n| < \infty$, there is a positive integer N such that

$$|a_{N+1}| + \cdots\cdots\cdots + |a_{N+p}| < \varepsilon \qquad \cdots\cdots\cdots (*)$$

for all $p = 1, 2....$ Notice that since σ is a bijection, we can find a positive integer M such that $M > \max\{j : 1 \leq \sigma(j) \leq N\}$. Then $\sigma(i) \geq N$ if $i \geq M$. This together with $(*)$ imply that if $i \geq M$ and $p \in \mathbb{N}$, we have

$$|a_{\sigma(i+1)}| + \cdots\cdots\cdots |a_{\sigma(i+p)}| < \varepsilon.$$

Thus the series $\sum_n a_{\sigma(n)}$ is absolutely convergent by the Cauchy criteria.
Finally we claim that $\sum_n a_n = \sum_n a_{\sigma(n)}$. Put $l = \sum_n a_n$ and $l' = \sum_n a_{\sigma(n)}$. Now let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that

$$|l - \sum_{n=1}^{N} a_n| < \varepsilon \quad \text{and} \quad |a_{N+1}| + \cdots\cdots + |a_{N+p}| < \varepsilon \cdots\cdots\cdots (**)$$

for all $p \in \mathbb{N}$. Now choose a positive integer M large enough so that $\{1, ..., N\} \subseteq \{\sigma(1), ..., \sigma(M)\}$ and $|l' - \sum_{i=1}^{M} a_{\sigma(i)}| < \varepsilon$. Notice that since we have $\{1, ..., N\} \subseteq \{\sigma(1), ..., \sigma(M)\}$, the condition $(**)$ gives

$$|\sum_{n=1}^{N} a_n - \sum_{i=1}^{M} a_{\sigma(i)}| \leq \sum_{N<i<\infty} |a_i| \leq \varepsilon.$$

We can now conclude that

$$|l - l'| \leq |l - \sum_{n=1}^{N} a_n| + |\sum_{n=1}^{N} a_n - \sum_{i=1}^{M} a_{\sigma(i)}| + |\sum_{i=1}^{M} a_{\sigma(i)} - l'| \leq 3\varepsilon.$$

The proof is complete. □

4. Power series

Throughout this section, let

$$f(x) = \sum_{i=0}^{\infty} a_i x^i \qquad \cdots\cdots\cdots\cdots (*)$$

denote a formal power series, where $a_i \in \mathbb{R}$.

Lemma 4.1. *Suppose that there is $c \in \mathbb{R}$ with $c \neq 0$ such that $f(c)$ is convergent. Then*

(i) : $f(x)$ is absolutely convergent for all x with $|x| < |c|$.
(ii) : f converges uniformly on $[-\eta, \eta]$ for any $0 < \eta < |c|$.

Proof. For Part (i), note that since $f(c)$ is convergent, then $\lim a_n c^n = 0$. So there is a positive integer N such that $|a_n c^n| \leq 1$ for all $n \geq N$. Now if we fix $|x| < |c|$, then $|x/c| < 1$. Therefore, we have

$$\sum_{n=1}^{\infty} |a_n||x^n| \leq \sum_{n=1}^{N-1} |a_n||x^n| + \sum_{n \geq N} |a_n c^n||x/c|^n \leq \sum_{n=1}^{N-1} |a_n||x^n| + \sum_{n \geq N} |x/c|^n < \infty.$$

So Part (i) follows.
Now for Part (ii), if we fix $0 < \eta < |c|$,then $|a_n x^n| \leq |a_n \eta|^n$ for all n and for all $x \in [-\eta, \eta]$. On the other hand, we have $\sum_n |a_n \eta^n| < \infty$ by Part (i). So f converges uniformly on $[-\eta, \eta]$ by the M-test. The proof is finished. □

Remark 4.2. *In Lemma 4.9(ii), notice that if $f(c)$ is convergent, it does not imply f converges uniformly on $[-c, c]$ in general.*
For example, $f(x) := 1 + \sum_{n=1}^{\infty} \frac{x^n}{n}$. Then $f(-1)$ is convergent but $f(1)$ is divergent.

Definition 4.3. *Call the set dom $f := \{x \in \mathbb{R} : f(c)$ is convergent $\}$ the domain of convergence of f for convenience. Let $0 \leq r := \sup\{|c| : c \in$ dom $f\} \leq \infty$. Then r is called the radius of convergence of f.*

Remark 4.4. *Notice that by Lemma 4.9, then the domain of convergence of f must be the interval with the end points $\pm r$ if $0 < r < \infty$.*
When $r = 0$, then dom $f = \{0\}$.
Finally, if $r = \infty$, then dom $f = \mathbb{R}$.

Example 4.5. *If* $f(x) = \sum_{n=0}^{\infty} n!x^n$*, then* $r = (0)$*. In fact, notice that if we fix a non-zero number* x *and consider* $\lim_n |(n+1)!x^{n+1}|/|n!x^n| = \infty$*, then by the ratio test* $f(x)$ *must be divergent for any* $x \neq 0$*. So* $r = 0$ *and dom* $f = (0)$*.*

Example 4.6. *Let* $f(x) = 1 + \sum_{n=1}^{\infty} x^n/n^n$*. Notice that we have* $\lim_n |x^n/n^n|^{1/n} = 0$ *for all* x*. So the root test implies that* $f(x)$ *is convergent for all* x *and then* $r = \infty$ *and dom* $f = \mathbb{R}$*.*

Example 4.7. *Let* $f(x) = 1 + \sum_{n=1}^{\infty} x^n/n$*. Then* $\lim_n |x^{n+1}/(n+1)| \cdot |n/x^n| = |x|$ *for all* $x \neq 0$*. So by the ration test, we see that if* $|x| < 1$*, then* $f(x)$ *is convergent and if* $|x| > 1$*, then* $f(x)$ *is divergent. So* $r = 1$*. Also, it is known that* $f(1)$ *is divergent but* $f(-1)$ *is divergent. Therefore, we have dom* $f = [-1, 1)$*.*

Example 4.8. *Let* $f(x) = \sum x^n/n^2$*. Then by using the same argument of Example 4.7, we have* $r = 1$*. On the other hand, it is known that* $f(\pm 1)$ *both are convergent. So dom* $f = [-1, 1]$*.*

Lemma 4.9. *With the notation as above, if* $r > 0$*, then* f *converges uniformly on* $(-\eta, \eta)$ *for any* $0 < \eta < r$*.*

Proof. It follows from Lemma 4.1 at once. □

Remark 4.10. *Note that the Example 4.7 shows us that* f *may not converge uniformly on* $(-r, r)$*. In fact let* f *be defined as in Example 4.7. Then* f *does not converges on* $(-1, 1)$*. In fact, if we let* $s_n(x) = \sum_{k=0}^{\infty} a_k x^k$*, then for any positive integer* n *and* $0 < x < 1$*, we have*

$$|s_{2n}(x) - s_n(x)| = \frac{x^{n+1}}{n+1} + \cdots\cdots + \frac{x^n}{2n}.$$

From this we see that if n *is fixed, then* $|s_{2n}(x) - s_n(x)| \to 1/2$ *as* $x \to 1-$*. So for each* n*, we can find* $0 < x < 1$ *such that* $|s_{2n}(x) - s_n(x)| > \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$*. Thus* f *does not converges uniformly on* $(-1, 1)$ *by the Cauchy Theorem.*

Proposition 4.11. *With the notation as above, let* $\ell = \overline{\lim}\ |a_n|^{1/n}$ *or* $\lim \dfrac{|a_{n+1}|}{|a_n|}$ *provided it exists. Then*

$$r = \begin{cases} \frac{1}{\ell} & \text{if } 0 < \ell < \infty; \\ 0 & \text{if } \ell = \infty; \\ \infty & \text{if } \ell = 0. \end{cases}$$

Proposition 4.12. *With the notation as above if* $0 < r \leq \infty$*, then* $f \in C^{\infty}(-r, r)$*. Moreover, the* k*-derivatives* $f^{(k)}(x) = \sum_{n \geq k} a_k n(n-1)(n-2) \cdots\cdots (n-k+1)x^{n-k}$ *for all* $x \in (-r, r)$*.*

Proof. Fix $c \in (-r, r)$. By Lemma 4.9, one can choose $0 < \eta < r$ such that $c \in (-\eta, \eta)$ and f converges uniformly on $(-\eta, \eta)$.
It needs to show that the k-derivatives $f^{(k)}(c)$ exists for all $k \geq 0$. Consider the case $k = 1$ first.
If we consider the series $\sum_{n=0}^{\infty} (a_n x^n)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$, then it also has the same radius r because $\lim_n |na_n|^{1/n} = \lim_n |a_n|^{1/n}$. This implies that the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges uniformly on $(-\eta, \eta)$. Therefore, the restriction $f|(-\eta, \eta)$ is differentiable. In particular, $f'(c)$ exists and $f'(c) = \sum_{n=1}^{\infty} n a_n c^{n-1}$.
So the result can be shown inductively on k. □

Proposition 4.13. *With the notation as above, suppose that $r > 0$. Then we have*

$$\int_0^x f(t)dt = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt = \sum_0^{\infty} \frac{1}{n+1} a_n x^{n+1}$$

for all $x \in (-r, r)$.

Proof. Fix $0 < x < r$. Then by Lemma 4.9 f converges uniformly on $[0, x]$. Since each term $a_n t^n$ is continuous, the result follows. □

Theorem 4.14. (**Abel**) : *With the notation as above, suppose that $0 < r$ and $f(r)$ (or $f(-r)$) exists. Then f is continuous at $x = r$ (resp. $x = -r$), that is $\lim_{x \to r-} f(x) = f(r)$.*

Proof. Note that by considering $f(-x)$, it suffices to show that the case $x = r$ holds.
Assume $r = 1$.
Notice that if f converges uniformly on $[0, 1]$, then f is continuous at $x = 1$ as desired.
Let $\varepsilon > 0$. Since $f(1)$ is convergent, then there is a positive integer such that

$$|a_{n+1} + \cdots\cdots\cdots + a_{n+p}| < \varepsilon$$

for $n \geq N$ and for all $p = 1, 2....$ Note that for $n \geq N$; $p = 1, 2...$ and $x \in [0, 1]$, we have

$$\begin{aligned} s_{n+p}(x) - s_n(x) = {} & a_{n+1}x^{n+1} + a_{n+2}x^{n+1} + a_{n+3}x^{n+1} + \cdots\cdots\cdots + a_{n+p}x^{n+1} \\ & + a_{n+2}(x^{n+2} - x^{n+1}) + a_{n+3}(x^{n+2} - x^{n+1}) + \cdots\cdots\cdots + a_{n+p}(x^{n+2} - x^{n+1}) \\ & + a_{n+3}(x^{n+3} - x^{n+2}) + \cdots\cdots\cdots + a_{n+p}(x^{n+3} - x^{n+2}) \\ & \vdots \\ & + a_{n+p}(x^{n+p} - x^{n+p-1}). \end{aligned} \tag{4.1}$$

Since $x \in [0, 1]$, $|x^{n+k+1} - x^{n+k}| = x^{n+k} - x^{n+k+1}$. So the Eq.4.1 implies that

$$|s_{n+p}(x) - s_n(x)| \leq \varepsilon(x_{n+1} + (x^{n+1} - x^{n+2}) + (x^{n+2} - x^{n+3}) + \cdots + (x^{n+p-1} - x^{n+p})) = \varepsilon(2x^{n+1} - x^{n+p}) \leq 2\varepsilon.$$

So f converges uniformly on $[0, 1]$ as desired.
Finally for the general case, we consider $g(x) := f(rx) = \sum_n a_n r^n x^n$. Note that $\lim_n |a_n r^n|^{1/n} = 1$ and $g(1) = f(r)$. Then by the case above,, we have shown that

$$f(r) = g(1) = \lim_{x \to 1-} g(x) = \lim_{x \to r-} f(x).$$

The proof is finished. □

Remark 4.15. *In Remark 4.10, we have seen that f may not converges uniformly on $(-r, r)$. However, in the proof of Abel's Theorem above, we have shown that if $f(\pm r)$ both exist, then f converges uniformly on $[-r, r]$ in this case.*

5. Real analytic functions

Proposition 5.1. *Let $f \in C^\infty(a,b)$ and $c \in (a,b)$. Then for any $x \in (a,b) \setminus \{c\}$ and for any $n \in \mathbb{N}$, there is $\xi = \xi(x,n)$ between c and x such that*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \int_c^x \frac{f^{(n+1)}(t)}{n!}(x-t)^n dt$$

Call $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ (may not be convergent) the Taylor series of f at c.

Proof. It is easy to prove by induction on n and the integration by part. □

Definition 5.2. *A real-valued function f defined on (a,b) is said to be real analytic if for each $c \in (a,b)$, one can find $\delta > 0$ and a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ such that*

$$f(x) = \sum_{k=0}^{\infty} a_k(x-c)^k \qquad \cdots\cdots\cdots (*)$$

for all $x \in (c-\delta, c+\delta) \subseteq (a,b)$.

Remark 5.3.

(i) : Concerning about the definition of a real analytic function f, the expression $(*)$ above is uniquely determined by f, that is, each coefficient a_k's is uniquely determined by f. In fact, by Proposition 4.12, we have seen that $f \in C^\infty(a,b)$ and

$$a_k = \frac{f^{(k)}(c)}{k!} \qquad \cdots\cdots\cdots (**)$$

for all $k = 0, 1, 2, ...$

(ii) : Although every real analytic function is C^∞, the following example shows that the converse does not hold.
Define a function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0; \\ 0 & \text{if } x = 0. \end{cases}$$

One can directly check that $f \in C^\infty(\mathbb{R})$ and $f^{(k)}(0) = 0$ for all $k = 0, 1, 2....$ So if f is real analytic, then there is $\delta > 0$ such that $a_k = 0$ for all k by the Eq.$(**)$ above and hence $f(x) \equiv 0$ for all $x \in (-\delta, \delta)$. It is absurd.

(iii) **Interesting Fact** : *Let D be an open disc in $\mathbb{C}$. A complex analytic function f on D is similarly defined as in the real case. However, we always have: f is complex analytic if and only if it is C^∞.*

Proposition 5.4. *Suppose that $f(x) := \sum_{k=0}^{\infty} a_k(x-c)^k$ is convergent on some open interval I centered at c, that is $I = (c-r, c+r)$ for some $r > 0$. Then f is analytic on I.*

Proof. We first note that $f \in C^\infty(I)$. By considering the translation $x - c$, we may assume that $c = 0$. Now fix $z \in I$. Now choose $\delta > 0$ such that $(z-\delta, z+\delta) \subseteq I$. We are going to show that

$$f(x) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z)}{j!}(x-z)^j.$$

for all $x \in (z-\delta, z+\delta)$.
Notice that $f(x)$ is absolutely convergent on I. This implies that

$$\begin{aligned} f(x) &= \sum_{k=0}^{\infty} a_k (x-z+z)^k \\ &= \sum_{k=0}^{\infty} a_k \sum_{j=0}^{k} \frac{k(k-1)\cdot\dots\cdot(k-j+1)}{j!}(x-z)^j z^{k-j} \\ &= \sum_{j=0}^{\infty} (\sum_{k\geq j} k(k-1)\cdot\dots\cdot(k-j+1) a_k z^{k-j}) \ \frac{(x-z)^j}{j!} \\ &= \sum_{j=0}^{\infty} \frac{f^{(j)}(z)}{j!}(x-z)^j \end{aligned}$$

for all $x \in (z-\delta, z+\delta)$. The proof is finished. ☐

Example 5.5. *Let $\alpha \in \mathbb{R}$. Recall that $(1+x)^\alpha$ is defined by $e^{\alpha \ln(1+x)}$ for $x > -1$.*
Now for each $k \in \mathbb{N}$, put

$$\binom{\alpha}{k} = \begin{cases} \frac{\alpha(\alpha-1)\cdot\dots\cdot(\alpha-k+1)}{k!} & \text{if } k \neq 0; \\ 1 & \text{if } x = 0. \end{cases}$$

Then

$$f(x) := (1+x)^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k$$

whenever $|x| < 1$.
Consequently, $f(x)$ is analytic on $(-1, 1)$.

Proof. Notice that $f^{(k)}(x) = \alpha(\alpha-1)\cdot\dots\cdot(\alpha-k+1)(1+x)^{\alpha-k}$ for $|x| < 1$.
Fix $|x| < 1$. Then by Proposition 5.1, for each positive integer n we have

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k + \int_0^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} dt$$

So by the mean value theorem for integrals, for each positive integer n, there is ξ_n between 0 and x such that

$$\int_0^x \frac{f^{(n)}(t)}{(n-1)!}(x-t)^{n-1} dt = \frac{f^{(n)}(\xi_n)}{(n-1)!}(x-\xi_n)^{n-1} x$$

Now write $\xi_n = \eta_n x$ for some $0 < \eta_n < 1$ and $R_n(x) := \dfrac{f^{(n)}(\xi_n)}{(n-1)!}(x-\xi_n)^{n-1} x$. Then

$$R_n(x) = (\alpha-n+1)\binom{\alpha}{n-1}(1+\eta_n x)^{\alpha-n}(x-\eta_n x)^{n-1} x = (\alpha-n+1)\binom{\alpha}{n-1} x^n (1+\eta_n x)^{\alpha-1} (\frac{1-\eta_n}{1+\eta_n x})^{n-1}.$$

We need to show that $R_n(x) \to 0$ as $n \to \infty$, that is the Taylor series of f centered at 0 converges to f. By the Ratio Test, it is easy to see that the series $\sum_{k=0}^{\infty} (\alpha-k+1)\binom{\alpha}{k} y^k$ is convergent as $|y| < 1$.
This tells us that the series $\lim_n |(\alpha-n+1)\binom{\alpha}{n} x^n| = 0$.
On the other hand, note that we always have $0 < 1-\eta_n < 1+\eta_n x$ for all n because $x > -1$. Thus, we

can now conclude that $R_n(x) \to 0$ as $|x| < 1$. The proof is finished. Finally the last assertion follows from Proposition 5.4 at once. The proof is complete. □

References

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